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Time–Dependent Spatial Price Equilibrium Problem: Existence and Stability Results for the Quantity Formulation Model

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Abstract. This paper concerns with the study of the time–dependent variational inequality associated to the spatial price equilibrium model related to the quantity formulation. In particular existence the - orems of the solution to the associated variational inequality and a stability analysis of the equilibrium pattern is reported.

Key words: Spatial price equilibrium, Stability analysis, Time-dependent variational inequalities

1. Introduction

In the last years some papers have been devoted to the study of the influence of the time on the equilibrium problems (see [1–4]). In fact, we cannot avoid to consider that each phenomenon of our economic and physical world is not stable with respect to the time and that our static models of equilibria are a first useful abstract approach.

Then in this paper we are concerned with the spatial price equilibrium problem in the case of the quantity formulation under the assumption that the data evolve in the time (see [5] and [6] for the static case). We shall prove that the time depending equilibrium conditions can be incorporated directly into a time depending variational inequality for which an existence theorem is provided in a suitable Lebesgue class of functions. This fact means that, in order to obtain existence results, regularity assumptions with respect to the time are not requested. The same variational inequality formulation is useful in order to perform a stability analysis of the equilibrium pattern.

2. Time Depending Quantity Formulation

Let us consider n supply markets P_i, $i=1,2,\dots n$ and m de -mand markets Q_j, $j=1,2,\dots m$ involved in the production and in the consumption respectively of a commodity during a period of time $[0,T]$, $T>0$. Let $g_i(t)$, $t\in[0,T]$,

$i=1,2,\dots,n$, denote the supply of the commodity associated with supply market i at the time $t\in[0,T]$ and let $p_i(t)$, $t\in[0,T]$, $i=1,2,\dots,n$, denote the supply price of the commodity associated with supply market i at the same time $t\in[0,T]$. Let $f_j(t)$, $t\in[0,T]$, $j=1,2,\dots,m$, denote the demand associated with the demand market j at the time $t\in[0,T]$ and let $q_j(t)$, $t\in[0,T]$, $j=1,2,\dots,m$, denote the demand price associated with the demand market j at the time $t\in[0,T]$. Let $x_{ij}(t)$, $t\in[0,T]$, $i=1,2,\dots,n$, $j=1,2,\dots,m$, denote the nonnegative commodity shipment between the supply and demand pair (P_i,Q_j) at the time $t\in[0,T]$ and let $c_{ij}(t)$, $t\in[0,T]$, $i=1,2,\dots,n$, $j=1,2,\dots,m$, denote the nonnegative unit transportation cost associated with trading the commodity between (P_i,Q_j) at the same time $t\in[0,T]$.

Assuming that we are not in presence of excesses on the supply and on the demand, the following feasibility conditions must hold for every i and j and a.e. in $[0,T]$:

$$g_i(t)=\sum_{j=1}^{m}x_{ij}(t), \tag{1}$$

$$f_j(t)=\sum_{i=1}^{n}x_{ij}(t). \tag{2}$$

Furthermore, assuming that capacity constraints are not present, the feasible vector $u(t)=(g(t),f(t),x(t))\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{nm}$ satisfies the condition

$$u(t)\geqslant 0 \text{ a.e. in } [0,T]. \tag{3}$$

Let us assume, for technical reasons, that the functional setting for the trajectories $u(t)$ is the reflexive Banach space:

$$L=L^2(0,T,\mathbb{R}^n)\times L^2(0,T,\mathbb{R}^m)\times L^2(0,T,\mathbb{R}^{nm}).$$

We recall that, if z is an element of the Euclidean space $\mathbb{R}^k$, we denote by $\|z\|_k$ the usual Euclidean norm; then, if $z(t)$ is an element of $L^2(0,T,\mathbb{R}^k)$, we set

$$\|z\|_{L^2(0,T,\mathbb{R}^k)}=\left(\int_0^T\|z(t)\|_k^2\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

We shall omit the index k and we shall denote $\|z\|_{L^2(0,T,\mathbb{R}^k)}$ briefly by $\|z\|_{L^2}$ if there is no possibility of confusion. If $(z_1,z_2,z_3)\in L^2(0,T,\mathbb{R}^{k_1})\times L^2(0,T,\mathbb{R}^{k_2})\times L^2(0,T,\mathbb{R}^{k_3})$, we set

$$\|(z_1,z_2,z_3)\|_{L^2}=\left(\|z_1\|^2_{L^2(0,T,\mathbb{R}^{k_1})}+\|z_2\|^2_{L^2(0,T,\mathbb{R}^{k_2})}+\|z_3\|^2_{L^2(0,T,\mathbb{R}^{k_3})}\right)^{\frac{1}{2}},$$

and in particular if $u=(g,f,x)\in L$

$$\|u\|_L=\Big(\|g\|^2_{L^2(0,T,\mathbb{R}^n)}+\|f\|^2_{L^2(0,T,\mathbb{R}^m)}+\|x\|^2_{L^2(0,T,\mathbb{R}^{nm})},\Big)^{\frac{1}{2}}.$$

Hence, taking into account (1), (2), (3), the set of feasible vectors $u(t)=(g(t),f(t),x(t))$ is

$$\mathbb{K}=\Bigg\{u=(g,f,x)\in L: u(t)\geqslant 0 \text{ a.e. in } [0,T],$$

$$g_i(t)=\sum_{j=1}^{m}x_{ij},\quad i=1,\dots,n, \tag{4}$$

$$f_j(t)=\sum_{i=1}^{n}x_{ij},\quad j=1,\dots,m, \text{ a.e. in } [0,T]\Bigg\}.$$

It is easily seen that $\mathbb{K}$ is a convex, closed, not bounded subset of the Hilbert space L. Furthermore, we are giving the mappings:

$$\begin{aligned} &p: L^2(0,T,\mathbb{R}^n_+)\to L^2(0,T,\mathbb{R}^n_+)\\ &q: L^2(0,T,\mathbb{R}^m_+)\to L^2(0,T,\mathbb{R}^m_+), \end{aligned} \tag{5}$$

which assign to each supply g the supply price $p(g)$ and to each demand f the demand price $q(f)$. Then the dynamic market equilibrium condition in the case of the quantity formulation takes the following form:

DEFINITION 1. *$u=(g,f,x)\in L$ is a dynamic market equilibrium if and only if for each $i=1,\dots,n$ and $j=1,\dots,m$ and a.e. in $[0,T]$ there holds*:

$$p_i(g(t))+c_{ij}(x(t))\begin{cases} =q_j(f(t)), \ \textit{if} \ x_{ij}(t)>0\\ \geqslant q_j(f(t)), \ \textit{if} \ x_{ij}(t)=0. \end{cases} \tag{6}$$

Condition (6) states that if there is trade between a pair (P_i,Q_j) at the time t, then the supply price at supply market P_i plus the transportation cost between the pair of markets at the same time t must be equal to the demand price at demand market Q_j at the time t; whereas if the supply price plus the transportation cost at the same time t exceeds the demand price at the time t, then there will be no shipment between the supply and demand market pair at the time t.

Let us set:

$$\langle v(u),u\rangle=\int_0^T\{p(g(t))\cdot g(t)-q(f(t))\cdot f(t)+c(x(t))\cdot x(t)\}\,\mathrm{dt}.$$

The following characterization in terms of a variational inequality of the equilibrium solution holds.

THEOREM 2.1. *$u^*=(g^*,f^*,x^*)\in L$ is a dynamic market equilibrium if and only if u is a solution to the variational inequality*:

$$\begin{aligned}&\langle v(u^*),u-u^*\rangle\\ &=\int_0^T\Big\{p(g^*(t))\cdot(g(t)-g^*(t))-q(f^*(t))\cdot(f(t)-f^*(t))\\ &\quad+c(x^*(t))\cdot(x(t)-x^*(t))\Big\}\mathrm{dt}\geqslant 0,\quad \forall u=(g,f,x)\in\mathbb{K}.\end{aligned} \tag{7}$$

Proof. Let $u^*\in\mathbb{K}$ be an equilibrium solution according to (6). Then, from (6), we have for every $u\in\mathbb{K}$ and a.e. in $[0,T]$:

$$\Big(p_i(g^*(t))-q_j(f^*(t))+c_{ij}(x^*(t))\Big)\big(x_{ij}(t)-x^*_{ij}(t)\big)\geqslant 0 \tag{8}$$

$i=1,\dots,n,\ j=1,\dots,m$. From (8), we derive

$$\sum_{i=1}^n p_i(g^*(t))\left(\sum_{j=1}^m x_{ij}(t)-\sum_{j=1}^m x^*_{ij}(t)\right)-\sum_{j=1}^m q_j(f^*(t))\left(\sum_{i=1}^n x_{ij}(t)-\sum_{i=1}^n x^*_{ij}(t)\right)$$

$$+\sum_{i=1}^n\sum_{j=1}^m c_{ij}(x^*(t))\big(x_{ij}(t)-x^*_{ij}(t)\big)\geqslant 0$$

a.e. in $[0,T]$. Taking into account (1) and (2), we get:

$$\begin{aligned}&\sum_{i=1}^n p_i(g^*(t))(g_i(t)-g^*_i(t))-\sum_{j=1}^m q_j(f^*(t))\big(f_j(t)-f^*_j(t)\big)\\ &\quad+\sum_{i=1}^n\sum_{j=1}^m c_{ij}(x^*(t))\big(x_{ij}(t)-x^*_{ij}(t)\big)\geqslant 0 \text{ a.e. in } [0,T]\end{aligned}$$

and hence (7).
Viceversa, let u^* be a solution to (7) and assume that (6) does not hold. Then there exists a pair of indexes (i^*,j^*) together with a set $E\subseteq[0,T]$ having positive measure such that

$$p_{i^*}(g^*(t))+c_{i^*j^*}(x^*(t))<q_{j^*}(f^*(t)) \text{ on } E. \tag{9}$$

Let us chose in (7)

$$x_{ij}(t)=x^*_{ij}(t) \text{ for } i\neq i^*,\ j\neq j^* \tag{10}$$

$$x_{i^*j^*}(t)\begin{cases}=x^*_{i^*j^*}(t) & \text{if } t\in[0,T]\setminus E\\ >x^*_{i^*j^*}(t) & \text{if } t\in E.\end{cases} \tag{11}$$

Taking into account (1), (2) and (10), we get

$$g_i(t)-g_i^*(t)=\sum_{j=1}^{m}x_{ij}^*(t)-\sum_{j=1}^{m}x_{ij}^*(t)=0 \quad i\neq i^* \tag{12}$$

$$f_j(t)-f_j^*(t)=\sum_{i=1}^{n}x_{ij}^*(t)-\sum_{i=1}^{m}x_{ij}^*(t)=0 \quad j\neq j^*. \tag{13}$$

Then, in virtue of (10), (11), (12) and (13), we have:

$$\begin{aligned}
\langle v(u^*), u-u^*\rangle &= \int_0^T \Big\{ p_{i^*}(g^*(t))(g_{i^*}(t)-g_{i^*}^*(t)) \\
&\quad -q_{j^*}(f^*(t))\big(f_{j^*}(t)-f_{j^*}^*(t)\big)-c_{i^*j^*}(x^*(t))\big(x_{i^*j^*}(t)-x_{i^*j^*}^*(t)\big)\Big\}\,\mathrm{dt} \\
&= \int_0^T \Big\{ p_{i^*}(g^*(t))\Big(\sum_{j=1}^{m}x_{i^*j}(t)-\sum_{j=1}^{m}x_{i^*j}^*(t)\Big) \\
&\quad -q_{j^*}(f^*(t))\Big(\sum_{i=1}^{n}x_{ij^*}(t)-\sum_{i=1}^{n}x_{ij^*}^*(t)\Big) \\
&\quad -c_{i^*j^*}(x^*)\big(x_{i^*j^*}(t)-x_{i^*j^*}^*(t)\big)\Big\}\,\mathrm{dt} \\
&= \int_E \big(p_{i^*}(g^*(t))-q_{j^*}(f^*(t))-c_{i^*j^*}(x^*(t))\big)\big(x_{i^*j^*}(t)-x_{i^*j^*}^*(t)\big)\,\mathrm{dt}.
\end{aligned} \tag{14}$$

In virtue of (9) and (11), we get:

$$\langle v(u^*), u-u^*\rangle < 0,$$

that is a contradiction. Therefore the requested equivalence is achieved.

Remark 1. In a next paper we will be concerned with an improved model of spatial price equilibrium problem, namely we will consider the presence of excesses of supply and demand and the presence of the capacity constraints on p_i, g_j, x_{ij}. In this new framework, conditions (1) and (2) become respectively:

$$g_i(t)=\sum_{j=1}^{m}x_{ij}(t)+s_i(t), \tag{15}$$

$$f_j(t)=\sum_{i=1}^{n}x_{ij}(t)+\tau_j(t), \tag{16}$$

where $s_i(t)$ denotes the supply excess for the supply market P_i at the time $t \in [0,T]$ and $\tau_j(t)$ denotes the demand excess for the demand market Q_j at the time $t \in [0,T]$. Furthermore the capacity constraints on $p_i(g)$, $q_j(f)$ and $c_{ij}(x)$ $i = 1,2\ldots,n$ and $j = 1,2,\ldots,m$ will be described in the following way for each $(g,f,x) \in L$:

$$\begin{aligned} \underline{p}(t) &\leqslant p(g(t)) \leqslant \overline{p}(t) \quad \text{a.e. in } [0,T], \\ \underline{q}(t) &\leqslant q(f(t)) \leqslant \overline{q}(t) \quad \text{a.e. in } [0,T], \\ \underline{x}(t) &\leqslant x(t) \leqslant \overline{x}(t) \quad \text{a.e. in } [0,T], \end{aligned} \tag{17}$$

In this way, if we denote by $\tilde{\mathcal{L}}$ the space:

$$\tilde{\mathcal{L}} = L^2(0,T,\mathbb{R}^n) \times L^2(0,T,\mathbb{R}^m) \times L^2(0,T,\mathbb{R}^{nm}) \times L^2(0,T,\mathbb{R}^n) \times L^2(0,T,\mathbb{R}^m),$$

the set of feasible vectors becomes:

$$\begin{aligned} \tilde{\mathbb{K}} = \Bigg\{ & \tilde{u} = (g,f,x,s,\tau) \in \tilde{\mathcal{L}} : \tilde{u} \geqslant 0, \\ & g_i(t) = \sum_{j=1}^{m} x_{ij}(t) + s_i(t),\ i = 1,\ldots,n, \\ & f_j(t) = \sum_{i=1}^{n} x_{ij}(t) + \tau_j(t)\ j = 1,\ldots,m \text{ a.e. in } [0,T] \Bigg\}. \end{aligned}$$

and the equilibrium conditions assume the following form:

DEFINITION 2. *$\tilde{u}^* = (g^*, f^*, x^*, s^*, \tau^*) \in \tilde{\mathbb{K}}$ is a dynamic market equilibrium if and only if for each $i = 1,\ldots,n$ and $j = 1,\ldots,m$ and a.e. in $[0,T]$ there hold*:

$$\begin{aligned} & \text{if } s_i^*(t) > 0, \text{ then } p_i(g^*(t)) = \underline{p}_i(t); \\ & \text{if } \underline{p}_i(t) < p_i(g^*(t)) \leqslant \overline{p}_i(t), \text{ then } s_i^*(t) = 0; \\ & \text{if } \tau_j^*(t) > 0, \text{ then } q_j(f^*(t)) = \overline{q}_j(t); \\ & \text{if } \underline{q}_j(t) < q_j(f^*(t)) \leqslant \overline{q}_j(t), \text{ then } \tau_j^*(t) = 0; \\ & p_i(g^*(t)) + c_{ij}(x^*(t)) \begin{cases} = q_j(f^*(t)) & \text{if } \underline{x}_{ij}(t) < x_{ij}^*(t) \leqslant \overline{x}_{ij}(t) \\ \geqslant q_j(f^*(t)) & \text{if } x_{ij}^*(t) = \underline{x}_{ij}(t). \end{cases} \end{aligned}$$

As a consequence, the variational inequality associated to this improved model is:

"Find $\tilde{u}^* = (g^*, f^*, x^*, s^*, \tau^*) \in \tilde{\mathbb{K}}$ such that

$$\begin{aligned}&\int_0^T \Big\{p[g^*(t)][g(t)-g^*(t)]-\underline{p}(t)[s(t)-s^*(t)]\\&-q[f^*(t)][f(t)-f^*(t)]+\overline{q}(t)[\tau(t)-\tau^*(t)]\\&+c[x^*(t)][x(t)-x^*(t)]\Big\}\mathrm{dt}\geqslant 0 \quad \forall u=(g,f,x,s,\tau)\in\tilde{\mathbb{K}}",\end{aligned}\tag{18}$$

which, taking into account conditions (15) and (16) can also be rewritten in the form:

$$\begin{aligned}&\int_0^T \Big\{p[x^*(t),s^*(t)][x(t)-x^*(t)+s(t)-s^*(t)]-\underline{p}(t)[s(t)-s^*(t)]\\&-q[x^*(t),\tau^*(t)][x(t)-x^*(t)+\tau(t)-\tau^*(t)]+\overline{q}(t)[\tau(t)-\tau^*(t)]\\&+c[x^*(t)][x(t)-x^*(t)]\Big\}\mathrm{dt}=\\&=\int_0^T \Big\{[p[x^*(t),s^*(t)]-q[x^*(t),\tau^*(t)]+c[x^*(t)]][x(t)-x^*(t)]\\&+\Big[p[x^*(t),s^*(t)]-\underline{p}(t)\Big][s(t)-s^*(t)]\\&-[q[x^*(t),\tau^*(t)]-\overline{q}(t)][\tau(t)-\tau^*(t)]\Big\}\mathrm{dt}\geqslant 0,\end{aligned}$$

$\forall (x,s,\tau)\in\mathbb{K}^1$, where

$$\mathbb{K}^1=\{(x,s,\tau)\in L: x(t)\geqslant 0,\ s(t)\geqslant 0,\ \tau(t)\geqslant 0 \text{ a.e. in } [0,T]\}.$$

3. Existence Theorems

Let us recall some concepts that will be useful in the following. Let E be a real topological vector space, $\mathbb{K}\subseteq E$ convex. Then $v:\mathbb{K}\to E^*$ is said to be:

1. *pseudomonotone* if and only if

$$\forall u_1,u_2\in\mathbb{K}\quad \langle v(u_1),u_2-u_1\rangle\geqslant 0\Rightarrow\langle v(u_2),u_1-u_2\rangle\leqslant 0;$$

2. *hemicontinuous* if and only if

$$\forall u\in\mathbb{K}\quad \text{the function}\quad z\to\langle v(z),u-z\rangle$$

is upper semicontinuous on $\mathbb{K}$;

3. *hemicontinuous along line segments* if and only if

$$\forall u_1,u_2\in\mathbb{K}\quad \text{the function}\quad z\to\langle v(z),u_2-u_1\rangle$$

is upper semicontinuous on the line segment $[u_1,u_2]$.

Adapting a classical existence theorem for the solution of a variational inequality to our problem, we will have the following theorem, which provide existence with or without pseudomonotonicity assumptions. Moreover, since the convex set $\mathbb{K}$ is unbounded, we need coercivity assumptions, which we use in a generalized version (see [6] for the usual coercivity condition).

THEOREM 3.1. *Each of the following conditions is sufficient to ensure the existence of the solution of (7)*:

1. $v(u)=v(g(t),f(t),x(t))=(p(g(t)),q(f(t)),c(x(t)))$ *is hemicontinuous with respect to the strong topology and there exist* $A\subseteq\mathbb{K}$ *compact and there exist* $B\subseteq\mathbb{K}$ *compact, convex with respect to the strong topology such that*
$$\forall u_1=(p_1,q_1,x_1)\in\mathbb{K}\setminus A \quad \exists u_2=(p_2,q_2,x_2)\in B$$
$$\text{such that } \langle v(u_1),u_2-u_1\rangle<0;$$
2. v *is pseudomonotone,* v *is hemicontinuous along line segments and there exist* $A\subseteq\mathbb{K}$ *compact and* $B\subseteq\mathbb{K}$ *compact, convex with respect to the weak topology such that*
$$\forall p\in\mathbb{K}\setminus A \quad \exists\tilde{p}\in B: \langle v(p),\tilde{p}-p\rangle<0;$$
3. v *is hemicontinuous on* $\mathbb{K}$ *with respect to the weak topology,* $\exists A\subseteq\mathbb{K}$ *compact,* $\exists B\subseteq\mathbb{K}$ *compact, convex with respect to the weak topology such that*
$$\forall p\in\mathbb{K}\setminus A \quad \exists\tilde{p}\in B: \langle v(p),\tilde{p}-p\rangle<0.$$

4. Stability Analysis of the Equilibrium Patterns

Let us consider the network model governed by the variational inequality (7). Let us assume that the supply price functions change from $p(\cdot)$ to $p^*(\cdot)$, the demand price functions change from $q(\cdot)$ to $q^*(\cdot)$ and the transportation cost functions change from $c(\cdot)$ to $c^*(\cdot)$ and let us establish the relation between the corresponding equilibrium patterns (g,f,x) and (g^*,f^*,x^*).

Let us require that the following strong monotonicity condition holds on $p(\cdot)$, $q(\cdot)$ and $c(\cdot)$:
$$\begin{aligned}&\int_0^T\Big\{\big[p(g^1(t)-p(g^2(t)))\big]\cdot\big[g^1(t)-g^2(t)\big]-\big[q(f^1(t))-q(f^2(t))\big]\cdot\\&\cdot\big[f^1(t)-f^2(t)\big]+\big[c(x^1(t))-c(x^2(t))\big]\cdot\big[x^1(t)-x^2(t)\big]\Big\}\mathrm{dt}\geqslant\\&\geqslant\alpha\big(\|g^1(t)-g^2(t)\|^2_{L^2}+\|f^1(t)-f^2(t)\|^2_{L^2}+\|x^1(t)-x^2(t)\|^2_{L^2}\big)\\&\forall(g^1,f^1,x^1),\quad(g^2,f^2,x^2)\in\mathbb{K}\quad\text{and }\alpha>0.\end{aligned}\tag{19}$$

A sufficient condition to ensure (19) is that $\forall (g^1, f^1, x^1) \in \mathbb{K}$, $(g^2, f^2, x^2) \in \mathbb{K}$:

$$\int_0^T \left[p(g^1(t) - p(g^2(t))) \right] \cdot \left[g^1(t) - g^2(t) \right] \mathrm{dt} \geqslant \beta \| g^1(t) - g^2(t) \|_{L^2}^2$$

$$-\int_0^T \left[q(f^1(t)) - q(f^2(t)) \right] \cdot \left[f^1(t) - f^2(t) \right] \mathrm{dt} \geqslant \gamma \| f^1(t) - f^2(t) \|_{L^2}^2 \tag{20}$$

$$\int_0^T \left[c(x^1(t)) - c(x^2(t)) \right] \cdot \left[x^1(t) - x^2(t) \right] \mathrm{dt} \geqslant \delta \| x^1(t) - x^2(t) \|_{L^2}^2$$

where $\beta, \gamma, \delta > 0$.

The following theorem establishes that small changes in the supply price, demand price and transportation cost functions induce small changes in the supplies, demands and commodity shipment patterns.

THEOREM 4.1. *Let α be a positive constant in the definition of strong monotonicity. Then*

$$\begin{aligned} &\| (g^*(t) - g(t)), (q^*(t) - q(t)), (x^*(t) - x(t)) \|_L \leqslant \frac{1}{\alpha} \\ &\cdot \| \left[(p^*(g^*(t)) - p(g^*(t))), -(q^*(f^*(t)) - q(f^*(t))), \right. \\ &\qquad \left. (c^*(x^*(t)) - c(x^*(t))) \right] \|_L . \end{aligned} \tag{21}$$

Proof. The vectors (g, f, x) and (g^*, f^*, x^*) satisfy the variational inequality (7), namely:

$$\begin{aligned} &\int_0^T \Big\{ p(g(t))(g'(t) - g(t)) - q(f(t))(f'(t) - f(t)) \\ &+ c(x(t))(x'(t) - x(t)) \Big\} \mathrm{dt} \geqslant 0, \quad \forall (g', f', x') \in \mathbb{K} \end{aligned} \tag{22}$$

$$\begin{aligned} &\int_0^T \Big\{ p^*(g^*(t))(g'(t) - g^*(t)) - q^*(f^*(t))(f'(t) - f^*(t)) \\ &+ c^*(x^*(t))(x'(t) - x^*(t)) \Big\} \mathrm{dt} \geqslant 0, \quad \forall (g', f', x') \in \mathbb{K}. \end{aligned} \tag{23}$$

Let us choose $g'(t) = g^*(t)$, $f'(t) = f^*(t)$, $x'(t) = x^*(t)$ in (22) and $g'(t) = g(t)$, $f'(t) = f(t)$, $x'(t) = x(t)$ in (23) and let us add the two resulting inequalities, then we obtain:

$$\begin{aligned} &\int_0^T \Big\{ [p^*(g^*(t)) - p(g(t))] \cdot [g(t) - g^*(t)] \\ &- [q^*(f^*(t)) - q(f(t))] \cdot [f(t) - f^*(t)] \\ &+ [c^*(x^*(t)) - c(x(t))] \cdot [x(t) - x^*(t)] \Big\} \mathrm{dt} \geqslant 0, \end{aligned} \tag{24}$$

that is

$$\int_0^T \Big\{[p^*(g^*(t))-p(g^*(t))+p(g^*(t))-p(g(t))]\cdot[g(t)-g^*(t)]$$
$$-[q^*(f^*(t))-q(f^*(t))+q(f^*(t))-q(f(t))]\cdot[f(t)-f^*(t)]+ \tag{25}$$
$$+[c^*(x^*(t))-c(x^*(t))+c(x^*(t))-c(x(t))]\cdot[x(t)-x^*(t)]\Big\}\,\mathrm{dt}\geqslant 0.$$

Using the monotonicity condition (19), (25) yields:

$$\int_0^T \Big\{[p^*(g^*(t))-p(g^*(t))]\cdot[g(t)-g^*(t)]$$
$$-[q^*(f^*(t))-q(f^*(t))]\cdot[f(t)-f^*(t)]$$
$$+[c^*(x^*(t))-c(x^*(t))]\cdot[x(t)-x^*(t)]\Big\}\,\mathrm{dt}$$
$$\geqslant \int_0^T \Big\{[p(g^*(t))-p(g(t))]\cdot[g^*(t)-g(t)] \tag{26}$$
$$-[q(f^*(t))-q(f(t))]\cdot[f^*(t)-f(t)]+$$
$$+[c(x^*(t))-c(x(t))]\cdot[x^*(t)-x(t)]\Big\}\,\mathrm{dt}$$
$$\geqslant \alpha\big(\|g^*(t)-g(t)\|^2_{L^2}+\|q^*(t)-q(t)\|^2_{L^2}+\|x^*(t)-x(t)\|^2_{L^2}\big).$$

Now let us observe that in virtue of the Schwarz inequality we have:

$$\int_0^T \Big\{[p^*(g^*(t))-p(g^*(t))]\cdot[g(t)-g^*(t)]$$
$$-[q^*(f^*(t))-q(f^*(t))]\cdot[f(t)-f^*(t)]$$
$$+[c^*(x^*(t))-c(x^*(t))]\cdot[x^*(t)-x(t)]\Big\}\,\mathrm{dt} \tag{27}$$
$$\leqslant \Big\{\|p^*(g^*(t))-p(g^*(t))\|^2_{L^2}+\|q^*(f^*(t))-q(f^*(t))\|^2_{L^2}$$
$$+\|c^*(x^*(t))-c(x^*(t))\|^2_{L^2}\Big\}^{\frac{1}{2}}$$
$$\cdot\big\{\|g(t)-g^*(t)\|^2_{L^2}+\|f(t)-f^*(t)\|^2_{L^2}+\|x(t)-x^*(t)\|^2_{L^2}\big\}^{\frac{1}{2}}.$$

Then from (27) we get

$$\alpha\big(\|g^*(t)-g(t)\|^2_{L^2}+\|f^*(t)-f(t)\|^2_{L^2}+\|x^*(t)-x(t))\|^2_{L^2}\big)^{\frac{1}{2}}$$
$$\leqslant \Big(\|p^*(g^*(t))-p(g^*(t))\|^2_{L^2}+\|q^*(f^*(t))-q(f^*(t))\|^2_{L^2} \tag{28}$$
$$+\|c^*(x^*(t))-c(x^*(t))\|^2_{L^2}\Big)^{\frac{1}{2}}.$$

THEOREM 4.2. *Let us consider the spatial price equilibrium problem with two supply price functions $p(\cdot)$ and $p^*(\cdot)$, two demand price functions $q(\cdot)$ and $q^*(\cdot)$ and two transportation cost functions $c(\cdot)$ and $c^*(\cdot)$ and let (g, f, x) and (g^*, f^*, x^*) be the corresponding equilibrium supply, demand and transportation patterns. Then the following estimates hold*:

$$\begin{aligned}\int_0^T \Big\{ &\sum_i [p_i^*(g^*(t)) - p_i(g(t))] \times (g_i^*(t) - g_i(t)) \\ &+ \sum_{i,j} \big[c_{ij}^*(x^*(t)) - c_{ij}(x(t))\big] \times (x_{ij}^*(t) - x_{ij}(t)) \\ &- \sum_j \big[q_j^*(f^*(t)) - q_j(f(t))\big] \times (f_j^*(t) - f_j(t)) \Big\} \mathrm{dt} \leqslant 0\end{aligned} \tag{29}$$

and

$$\begin{aligned}\int_0^T \Big\{ &\sum_i [p_i^*(g^*(t)) - p_i(g^*(t))] \times (g_i^*(t) - g_i(t)) \\ &+ \sum_{i,j} \big[c_{ij}^*(x^*(t)) - c_{ij}(x^*(t))\big] \times (x_{ij}^*(t) - x_{ij}(t)) \\ &- \sum_j \big[q_j^*(f^*(t)) - q_j(f^*(t))\big] \times (f_j^*(t) - f_j(t)) \Big\} \mathrm{dt} \leqslant 0\end{aligned} \tag{30}$$

Proof. Estimate (29) derives from (24), using the opposite sign. Estimate (30) derives from (25), using the opposite sign.

COROLLARY 1. *Let us assume that in the supply market i the supply price is increased (decreased), while all the other supply price functions remain fixed, that is: $p_i^*(g'(t)) \geqslant p_i(g'(t))$, $(p_i^*(g'(t)) \leqslant p_i(g'(t)))$ a.e. in $[0,T]$ for some i and $g'(t) \in \mathbb{K}$, while $p_j^*(g'(t)) = p_j(g'(t))$ for $j \neq i$ and $g' \in \mathbb{K}$. Let us assume also that $\dfrac{\partial p_j(g'(t))}{\partial g_i} = 0$ a.e. in $[0,T]$ for all $j \neq i$. If we fix the demand functions for all markets, that is $q_j^*(f'(t)) = q_j(f'(t))$ a.e. in $[0,T]$ for all j and for all $f'(t) \in \mathbb{K}$ and the transportation cost functions, that is $c_{ij}^*(x'(t)) = c_{ij}(x'(t))$ for all i, j and $x'(t) \in \mathbb{K}$, then there exists a set $E \subseteq \mathbb{K}$ such that $m(E) > 0$, $g_i^*(t) \leqslant g_i(t)$ and $p_i^*(g^*(t)) \geqslant p_i(g(t))$ in E.*

Proof. From (25) and taking into account (19), it follows that

$$\begin{aligned}\int_0^T \Big\{ &[p_i^*(g^*(t)) - p_i(g^*(t))] \cdot (g_i^*(t) - g_i(t)) \\ &+ \sum_{j \neq i} \big[p_j^*(g^*(t)) - p_j(g^*(t))\big] \cdot (g_j^*(t) - g_j(t)) \Big\} \mathrm{dt} \leqslant 0,\end{aligned}$$

that is

$$\int_0^T \left\{[p_i^*(g^*(t)) - p_i(g^*(t))] \cdot (g_i^*(t) - g_i(t))\right\} \mathrm{dt} \leqslant 0,$$

since $p_j^*(g^*(t)) = p_j(g^*(t))$ by assumption. Moreover, by assumption it holds $p_i^*(g'(t)) \geqslant p_i(g'(t))$, then there exists a subset $E \subseteq \mathbb{K}$ such that $m(E) > 0$ and $g_i^*(t) \leqslant g_i(t)$ in E.

Let us prove now that $p_i^*(g^*(t)) \geqslant p_i(g(t))$. From (23) and (22), it follows

$$\begin{aligned}\int_0^T \Big\{&[p_i^*(g^*(t)) - p_i(g(t))] \cdot (g_i^*(t) - g_i(t)) \\ &+ \sum_{j \neq i} \big[p_j^*(g^*(t)) - p_j(g(t))\big] \cdot (g_j^*(t) - g_j(t))\Big\} \mathrm{dt} \leqslant 0.\end{aligned} \tag{31}$$

Since $p_j^*(g^*(t)) = p_j(g^*(t))$ by assumption, (31) becomes:

$$\begin{aligned}\int_0^T \Big\{&[p_i^*(g^*(t)) - p_i(g(t))] \cdot (g_i^*(t) - g_i(t)) \\ &+ \sum_{j \neq i} \big[p_j(g^*(t)) - p_j(g(t))\big] \cdot (g_j^*(t) - g_j(t))\Big\} \mathrm{dt} \leqslant 0.\end{aligned} \tag{32}$$

Let us apply the Lagrange theorem to $p_j(g(t))$ in the interval $[g(t), g^*(t)]$ and (32) becomes:

$$\begin{aligned}\int_0^T \Big\{&[p_i^*(g^*(t)) - p_i(g(t))] \cdot (g_i^*(t) - g_i(t)) \\ &+ \sum_{j \neq i} \sum_{k=1}^{n} \frac{\partial p_j(\overline{g}(t))}{\partial g_k} \cdot (g_k^*(t) - g_k(t)) \cdot (g_j^*(t) - g_j(t))\Big\} \mathrm{dt} \leqslant 0.\end{aligned} \tag{33}$$

Since $\dfrac{\partial p_j(\overline{g}(t))}{\partial g_k} = 0$ by assumption, then from (33) it follows

$$\int_0^T \left\{[p_i^*(g^*(t)) - p_i(g(t))] \cdot (g_i^*(t) - g_i(t))\right\} \mathrm{dt} \leqslant 0$$

and hence $p_i^*(g^*(t)) \geqslant p_i(g(t))$ in E, since $g_j^*(t) - g_j(t) \leqslant 0$ by assumption.

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